

STABILITY PROPERTIES OF THE COLORED JONES POLYNOMIAL

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ABSTRACT. It is known that the colored Jones polynomial of an A -adequate link has a well-defined tail consisting of stable coefficients, and that the coefficients of the tail carry geometric and topological information of an A -adequate link complement. We use the ribbon graph expansion of the Kauffman bracket to show that a tail of the colored Jones polynomial can be defined for *all* links, and that it is constant if and only if the link is non A -adequate.

1. INTRODUCTION

For an oriented link $K \subset S^3$, the *colored Jones polynomial* is a sequence of Laurent polynomials $\{J_K(n, q)\}_{n=2}^{\infty}$, where $J_K(n, q) \in \mathbb{Z}[q^{-1/2}, q^{1/2}]$ is an invariant of K for each n . In particular, $J_K(2, q)$ is the ordinary Jones polynomial. Unlike the Alexander polynomial, which is defined based on the topology of the link complement, the colored Jones polynomial originated from representation theory, and its definition makes no reference to the topology of the link. Understanding the geometric and topological information that the Jones polynomial and the colored Jones polynomial carry has been a fundamental goal in the study of knots, 3-manifolds, and quantum invariants.

The Jones polynomial and the colored Jones polynomial of a link can be defined and studied in terms of combinatorial properties of link diagrams. One approach to this is through the Kauffman bracket construction [21]. Studying the combinatorics of the Kauffman bracket, Lickorish and Thistlethwaite [23] showed that the extreme degrees of the Jones polynomial are bounded by concrete data from any diagram. If the diagram is *semi-adequate*, which means that it is A -adequate or B -adequate (see Definition 2.2) then the bounds are sharp. The colored Jones polynomial of the class of semi-adequate links, which are links admitting a semi-adequate diagram, have since been studied considerably [27, 2, 1, 3, 6, 17, 15, 14], and they have been shown to relate to the topology and the geometry of the link complement [7, 8, 9, 11, 10]. Since a diagram is B -adequate if the mirror image of the diagram is A -adequate, for the rest of the paper we will only deal with A -adequacy. For the results discussed for A -adequate links, analogous statements for B -adequate links may be obtained by taking the mirror image of the diagram.

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Let $d(n)$ denote the minimum degree in q of $J_K(n, q)$. The formula for $J_K(n, q)$ based on the Kauffman bracket gives a lower bound $h_n(D)$ for $d(n)$, which can be explicitly computed from a diagram D of K , see Definition 2.3. If D is A -adequate, then $d(n) = h_n(D)$ [8].

Let

$$js_K := \left\{ \frac{4d(n)}{n^2} : n \geq 0 \right\}.$$

Garoufalidis' slope conjecture predicts that for each knot K , every limit point of js_K is a boundary slope of K . In [8], Futer, Kalfagianni, and Purcell use the fact that $h_n(D) = d(n)$ for links admitting an A -adequate diagram D and results [25, 9] on incompressible surfaces in the complement of A -adequate knots to prove the slope conjecture for the class of A -adequate knots.

It is also known that the coefficients of $J_K(n, q)$ exhibit a stability behavior for semi-adequate links. The stable coefficients directly relate to the geometry of hyperbolic link complements: For $i \geq 2$, let β_i be the coefficient of $q^{d(i)+(i-2)}$ of $J_K(i, q)$. Armond [1] and Garoufalidis and Le [14] have independently shown that for all $n \geq i$, the coefficient of $q^{d(n)+i-2}$ of $J_K(n, q)$ is equal to β_i if K is A -adequate. That is, the last $n - 1$ coefficients of $J_K(n, q)$ from $h_n(D)$ are *stable* for a link with an A -adequate diagram D , extending the results by Dasbach and Lin [6] on the last and penultimate stable coefficients of $J_K(n, q)$. For an A -adequate link K , they define the *stable coefficients* β_i and a power series

$$T_K(q) = \sum_{i=2}^{\infty} \beta_i q^i,$$

called a *tail* of the colored Jones polynomial. The last and penultimate coefficients of $J_K(n, q)$, and hence the last two coefficients of the tail, have been shown by the work of Futer, Kalfagianni, and Purcell to carry information on the geometric structure of the complement of an A -adequate knot, and to provide sharp volume bounds on the complement of hyperbolic links. See [9] for the results and a detailed survey.

Rozansky has shown that stability behavior also occurs in the categorification of the colored Jones polynomial [26]. The idea of this theory, developed independently by Frenkel, Stroppel and Sussan, Cooper and Krushkal, and Rozansky, is as follows: Given a link K , for each n , one assigns a chain complex $C^{Kh}(K, n)$ for which the graded Euler characteristic of its homology groups $H^{Kh}(K, n)$ is the n -th *colored Jones polynomial* $J_K(n, q)$. In [26], Rozansky shows that for K an A -adequate link, one can define a family of maps

$$f_n : \tilde{H}^{Kh}(K, n) \rightarrow \tilde{H}^{Kh}(K, n+1),$$

such that f_n is an isomorphism on \tilde{H}_i^{Kh} for $i \leq n - 1$, where i is the grading of the chain complex coming from the number of state circles of a Kauffman state. The tilde indicates the appropriate degree shifts made for the chain complex so that f_n is a degree-preserving map for each n . In other words, for all $n > i$, the homology groups of $C^{Kh}(K, n)$ of grading less than $i - 1$ are the same as the

homology groups of $C^{Kh}(K, i)$ of grading less than $i - 1$. As a result, he defines a *tail homology* of the colored Jones polynomial which is the direct limit of the direct system of $\tilde{H}^{Kh}(K_n)$ determined by f_n . The tail homology contains the *stable* homology groups of $C^{Kh}(K, n)$ as n increases. The last $n - 1$ coefficients of the graded Euler characteristic of the tail homology agree with the last $n - 1$ coefficients of $T_K(q)$ (hence the last $n - 1$ stable coefficients of $J_K(n, q)$). Therefore, we have a categorification of the tail in the case of A -adequate links, extending the results by Armond [1] and Garoufalidis and Le [14].

It is natural to ask whether similar results can be obtained outside the class of semi-adequate links. Manchon [24] has constructed an infinite family of non A -adequate knots with diagrams D for which $d(2) = h_2(D)$. An example of such a knot is 12n706, see [19] for a diagram of the knot. For this knot, $d(2) = h_2(D) = -4$. However, this knot is not A -adequate since the last coefficient of its Jones polynomial is 2, and it is known that for an A -adequate link, the last coefficient of its Jones polynomial is ± 1 [6]. Manchon's examples show that the degree of the Jones polynomial is not enough to characterize links which are not A -adequate. In general, Garoufalidis and Le [13] have shown that the colored Jones polynomial for every link satisfies a recurrence relation. In other words, $J_K(n, q)$ for n large enough is determined by finitely many initial terms in the sequence. This can be viewed as a weaker form of stability.

In this paper, we study the effect that a diagram being *non* A -adequate has on the last $n - 2$ coefficients of $J_K(n, q)$. Recall that $d(n)$ is the minimum degree of $J_K(n, q)$ and $h_n(D)$ is a lower bound of $d(n)$ obtained from a diagram D of K . It was shown by Kalfagianni and the author [20], that $d(n) > h_n(D)$ for $n > 2$ if D is not A -adequate. Therefore the colored Jones polynomial can characterize A -adequate links. Our main result, which extends that of [20], is the following:

Theorem 1.1. *Let K be a link with diagram D . If D is not A -adequate, then $d(n) \geq h_n(D) + n - 2$ for $n > 2$.*

If K is not A -adequate, then any link diagram $D = D(K)$ will not be A -adequate. Let h_n be the maximum of $h_n(D)$ taken over all diagrams D of K . By Theorem 1.1, $d(n) \geq h_n + n - 2$ for $n > 2$. We use this to obtain a link invariant in the form of a power series $J_K^A(q)$, which vanishes if K is not A -adequate as in [20]. This allows us to extend the construction of a tail of an A -adequate link to all links. See the discussion in Section 4.

Rozansky conjectures [26, Conjecture 2.13] that the tail homology vanishes for non A -adequate links and thus the last $n - 2$ -coefficients of the n th-colored Jones polynomial of a non A -adequate link should be identically 0. Theorem 1.1 gives an affirmative answer to the part of Rozansky's conjecture concerning the last $n - 2$ coefficients of the n th-colored Jones polynomial.

To describe the organization of the current paper, we recall that the degree lower bound $h_n(D)$, mentioned above, comes from an upper bound $M(D^n)$ of the maximum degree of the Kauffman bracket of the n -cable D^n of D . In Section 2.1, we review the definition of $J_K(n, q)$ in terms of $\langle D^n \rangle$, define the quantities $h_n(D)$ and

$M(D^n)$, and show that Theorem 1.1 is an immediate corollary of Theorem 2.4. Let $d^*\langle D^n \rangle$ denote the maximum degree of $\langle D^n \rangle$ in A . Theorem 2.4 states that $M(D^n) - d^*\langle D^n \rangle \geq 4(n-2)$ when D is not A -adequate. Since $h_n(D)$ is obtained from $M(D^n)$ by adding and multiplying by constants, Theorem 1.1 can be reduced to showing Theorem 2.4. In Section 2.2, we recall how to obtain a ribbon graph from a link diagram and review the *spanning sub-graph expansion* of the Kauffman bracket in terms of ribbon graphs. We use this setup to prove Theorem 2.4 in Section 3. To make the idea of our proof clearer to the reader, in Section 5 we illustrate the main points with a worked out example. In Section 4, using Theorem 1.1, we extract a link invariant out of $J_K^A(q)$ that detects precisely when a link is A -adequate. Finally, we discuss some applications of our construction.

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2. BACKGROUND

2.1. The colored Jones polynomial and the Kauffman bracket.

We give needed definitions in order to state Theorem 1.1 and Theorem 2.4. We also show that Theorem 2.4 implies Theorem 1.1. For the rest of this section, K is an oriented link and D is a diagram of K . The colored Jones polynomial has a convenient expression in terms of the Kauffman bracket of the Chebyshev polynomials of a diagram D of K . See [22, Chapter 13] for more details and references.

Definition 2.1. The *Chebyshev polynomials* $S_n(D)$ for $n = 0, 1, 2, \dots$ are defined by the following recurrence relation:

$$S_{n+1}(D) = DS_n(D) - S_{n-1}(D), \quad S_1(D) = D, \quad S_0(D) = 1,$$

where $D \cdot D \cdots D$ (k -times) is interpreted to be D^k , the link diagram obtained from D by taking k parallel copies of D . Here the convention is that $D^1 = D$.

Let $\omega(D)$ be the writhe of D , the *reduced $n+1$ -colored Jones polynomial of K* , denoted by $J_K(n+1, q)$, is obtained by substituting $q := A^{-4}$ into the polynomial

$$G_D(n+1, A) := \left((-1)^n A^{n^2+2n} \right)^{-\omega(D)} (-1)^{n-1} \left(\frac{A^4 - A^{-4}}{A^{2n} - A^{-2n}} \right) \langle S_n(D) \rangle,$$

where $\langle S_n(D) \rangle$ is the Kauffman bracket extended linearly over $S_n(D)$. Let $d^*\langle S_n(D) \rangle$ denote the maximum degree in A of $\langle S_n(D) \rangle$. From this formula, we see that $d(n)$, the minimum degree of $J_K(n, q)$ in q , is given by:

$$d(n) = -\frac{1}{4}(d^*\langle S_{n-1}(D) \rangle + 4 - 2(n-1) - \omega(D)((n-1)^2 + 2(n-1))).$$

We review the notion of A -adequacy here, which we will use to obtain an upper bound of $d^*\langle S_{n-1}(D) \rangle$. Let D be an oriented link diagram and x a crossing of D . Associated to D and x are two link diagrams, each with one fewer crossing than D , called the A -resolution and B -resolution of the crossing, see Figure 1.

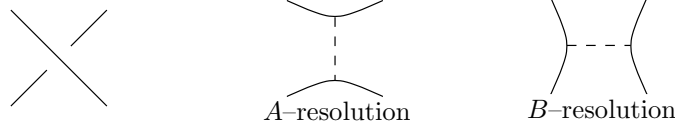


FIGURE 1. A - and B -resolutions of a crossing.

A *Kauffman state* σ of D is a choice of A -resolution or B -resolution at each crossing of D . Applying a Kauffman state to D , we obtain a crossing-free diagram consisting of a disjoint collection of simple closed curves on the projection plane. We call these curves *state circles*. The *all- A* state chooses the A -resolution at each crossing of D . We denote the union of the corresponding state circles by $s_A(D)$, and let $|s_A(D)|$ be the number of state circles. For an arbitrary state we similarly denote the union of the corresponding state circles by $s_\sigma(D)$ and let $|s_\sigma(D)|$ be the number of state circles in the union.



FIGURE 2. The figure 8 knot, its all A -resolution and the resulting all- A state graph.

Definition 2.2. To the union of state circles $s_\sigma(D)$ resulting from applying a Kauffman state σ , we attach one edge for each crossing which records the original location of the crossing on D . (These edges are dashed in Figure 1.) We will call the union of the state circles and the edges *the graph of the σ -resolution*, or the *σ -state graph*, and denote it by $H_\sigma(D)$. If σ is the all- A state then we call it the *graph of the A -resolution*, or the *all- A state graph* and denote it by $H_A(D)$. A diagram D is *A -adequate* if $H_A(D)$ does not contain any edges with both ends on the same state circle. Whenever it is clear which diagram we are referring to, we will just write H_σ for $H_\sigma(D)$ and H_A for $H_A(D)$ to simplify notation.

Definition 2.3. Let $c(D)$ be the number of crossings of D , $\omega(D)$ be the writhe of D , and $|s_A(D)|$ be the number of state circles in the all A -state of D . We define

$$M(D) := c(D) + 2|s_A(D)| - 2,$$

and

$$h_n(D) := -\frac{1}{4}(M(D^{n-1}) + 4 - 2(n-1) - \omega(D)((n-1)^2 + 2(n-1))).$$

It is known that for any link diagram D , the maximum degree $d^*\langle D \rangle$ of $\langle D \rangle$ in A is less than or equal to $M(D)$. Moreover, $d^*\langle D \rangle = M(D)$ if D is A -adequate [23].

An important corollary of the recursive formula for $S_n(D)$ is that

$$\langle S_n(D) \rangle = \langle D^n \rangle + (1-n)\langle D^{n-2} \rangle + \langle \text{lower degree cablings of } D \rangle.$$

Therefore, $d^*\langle S_n(D) \rangle \leq M(D^n)$ and $d(n) \geq h_n(D)$ for all $n > 2$. This shows that $h_n(D)$ is a lower bound for $d(n)$. If D is A -adequate, then D^n is also A -adequate and thus $h_n(D) = d(n)$ [8].

The main theorem that we prove in this paper is the following:

Theorem 2.4. *Let D be a link diagram that is not A -adequate. Then for $n > 2$.*

$$d^*\langle D^n \rangle \leq M(D^n) - 4(n-1).$$

We show that Theorem 2.4 implies Theorem 1.1:

Proof. The n -cable D^n has $n^2c(D)$ crossings and $n|s_A(D)|$ number of state circles in the all- A state, therefore

$$M(D^n) = n^2c(D) + 2n|s_A(D)| - 2.$$

Note that

$$M(D^n) - M(D^{n-2}) = c(D)(n^2 - (n-2)^2) + |s_A(D)|(2n - 2(n-2)) > 4(n-1).$$

Assuming Theorem 2.4, so $M(D^n) - d^*\langle D^n \rangle \geq 4(n-1)$, then $M(D^n) - d^*\langle S_n(D) \rangle \geq 4(n-1)$. Plugging this back into the expressions for $d(n)$ and $h_n(D)$ gives us that $d(n) \geq h_n(D) + n - 2$. \square

2.2. Ribbon graphs and the Kauffman bracket.

The Kauffman bracket of any diagram can be computed by means of ribbon graph expansions. As defined in [5], a *ribbon graph* is a multi-graph (i.e. loops and multiple edges are allowed) equipped with a cyclic order on the edges at every vertex. A ribbon graph can be embedded on an orientable surface such that every region in the complement of the graph is a disk. We called the regions the *faces* of the ribbon graph. For a ribbon graph \mathbb{G} we define the following combinatorial quantities:

$v(\mathbb{G})$ = the number of vertices of \mathbb{G} .

$e(\mathbb{G})$ = the number of edges of \mathbb{G} .

$f(\mathbb{G})$ = the number of faces of \mathbb{G} .

$k(\mathbb{G})$ = the number of connected components of \mathbb{G} .

$g(\mathbb{G}) = \frac{2k(\mathbb{G}) - v(\mathbb{G}) + e(\mathbb{G}) - f(\mathbb{G})}{2}$, the *genus* of \mathbb{G} .

For each Kauffman state σ of a link diagram, a ribbon graph \mathbb{G}_σ is constructed as follows: We orient the collection of non-intersecting state circles $s_\sigma(D)$ according to whether a circle is inside an odd or even number of circles, respectively. The vertices of \mathbb{G}_σ correspond to the collection of state circles and the edges to the crossings. The orientation of the circles defines the orientation of the edges around vertices. An equivalent definition of an A -adequate diagram is to say that a diagram is A -adequate if the ribbon graph \mathbb{G}_A constructed from the all- A state of the diagram has no one-edge loops. For more details, see [4], where the word *dessin* is used instead of ribbon graph.

Definition 2.5. A *spanning sub-graph* $\mathbb{H} \subseteq \mathbb{G}_A$ is a ribbon graph obtained by removing edges from \mathbb{G}_A .

From the ribbon graph setting we obtain the *spanning sub-graph expansion* of the Kauffman bracket, originally introduced in [4] and proven in [5].

Theorem 2.6. Let \mathbb{G}_A be the ribbon graph constructed from the all- A state of a link diagram D . Then

$$\langle D \rangle = \sum_{\mathbb{H} \subseteq \mathbb{G}_A} A^{e(\mathbb{G}_A) - 2e(\mathbb{H})} (-A^2 - A^{-2})^{f(\mathbb{H}) - 1},$$

where \mathbb{H} ranges over all spanning sub-graphs of \mathbb{G}_A .

The term

$$X_{\mathbb{H}} := A^{e(\mathbb{G}_A) - 2e(\mathbb{H})} (-A^2 - A^{-2})^{f(\mathbb{H}) - 1}$$

is the contribution of a spanning sub-graph \mathbb{H} to $\langle D \rangle$. Let the *first sth-coefficient* of $\langle D \rangle$ from $M(D)$ be the coefficient of $A^{M(D) - 4(s-1)}$ in $\langle D \rangle$. A spanning sub-graph \mathbb{H} contributes to the first sth coefficient of $\langle D \rangle$ if and only if $v(\mathbb{H}) - k(\mathbb{H}) + g(\mathbb{H}) \leq s - 1$, see [5, Theorem 6.1]. This is used in [5] to give a formula of the penultimate coefficient of $J_K(n, q)$ from $h_n(D)$ when K has an A -adequate diagram. Theorem 2.4 can be rephrased as saying that the first $n - 1$ coefficients of $\langle D^n \rangle$ from $M(D^n)$ are equal to zero. Our proof will depend heavily on Theorem 2.6 and the restriction on $v(\mathbb{H}) - k(\mathbb{H})$ for a spanning sub-graph \mathbb{H} that contributes to the first $n - 1$ coefficients of $\langle D^n \rangle$.

Definition 2.7. Let $\mathbb{H} \subseteq \mathbb{G}_A$ be a spanning sub-graph and σ be the Kauffman state that chooses the B -resolution on a crossing corresponding to an edge in \mathbb{H} and the A -resolution on a crossing corresponding to an edge not in \mathbb{H} . We associate to \mathbb{H} the state graph H of σ .

The following important lemma counts the faces of a spanning sub-graph by the number of state circles in its associate state graph.

Lemma 2.8. The number of state circles in H is equal to the number of faces of \mathbb{H} .

Proof. Let σ be a Kauffman state of a link diagram D and $\hat{\sigma}$ be its dual state which resolves each crossing of the link the opposite way from σ . This means that if σ

chooses the A -resolution at one crossing, then $\hat{\sigma}$ chooses the B -resolution at that crossing, and vice versa. The lemma follows from [4, Lemma 4.2], which says that $f(G_\sigma) = |s_{\hat{\sigma}} D|$, where G_σ is a ribbon graph constructed from σ . \square

3. PROOF OF THEOREM 2.4

A crossing of a link diagram is nugatory if there is a closed curve in the projection plane meeting the diagram transversely at that crossing and does not intersect the diagram anywhere else. A link diagram is *reduced* if it contains no nugatory crossings. Throughout this section, D denotes a reduced, non A -adequate link diagram, D^n is its n -cable, and \mathbb{G}_A^n is the ribbon graph constructed as in Section 2.2 from the all- A state of D^n . A spanning sub-graph $\mathbb{H} \subseteq \mathbb{G}_A^n$ is obtained from \mathbb{G}_A^n by deleting edges, with the associated state graph H as in Definition 2.7. We will mainly be interested in the all- A state graph H_A^n associated to the spanning sub-graph of no edges.

By Theorem 2.6, we have

$$\langle D^n \rangle = \sum_{\mathbb{H} \subseteq \mathbb{G}_A^n} A^{e(\mathbb{G}_A^n) - 2e(\mathbb{H})} (-A^2 - A^{-2})^{f(\mathbb{H}) - 1},$$

where \mathbb{H} ranges through all spanning sub-graphs of \mathbb{G}_A^n . Recall that

$$X_{\mathbb{H}} = A^{e(\mathbb{G}_A^n) - 2e(\mathbb{H})} (-A^2 - A^{-2})^{f(\mathbb{H}) - 1}$$

is the contribution of \mathbb{H} to $\langle D^n \rangle$. We will first prove a lemma that allow us to sum the terms $X_{\mathbb{H}}$ in a certain way to show the cancellation of the first $n - 1$ coefficients of $\langle D^n \rangle$.

3.1. A cancellation lemma.

Lemma 3.1. *Let c, d be integers and $\delta = (-A^2 - A^{-2})$, the maximum degree of the sum*

$$\sum_{i=0}^k \binom{k}{i} A^{c-2i} \delta^{d+i}$$

in A is less than or equal to $c + 2d - 4k$.

Proof.

We factor out $A^c \delta^d$ to obtain

$$A^c \delta^d \sum_{i=0}^k \binom{k}{i} A^{-2i} \delta^i.$$

Note first that $A^{-2i}\delta^i$ has the same degree 0 for each i , and each product $A^{-2i}\delta^i$ only has non-zero coefficients for terms of degree A^{-4j} for j a nonnegative integer. For $j > 0$, the j th coefficient of the polynomial $A^c\delta^d \sum_{i=0}^k \binom{k}{i} A^{-2i}\delta^i$ of the term $A^{c+2d-4(j-1)}$ is given by

$$\text{the } j\text{th coefficient} = \sum_{i \geq j-1}^k (-1)^i \binom{k}{i} \binom{i}{j-1}.$$

To prove the lemma, it suffices to show that the j th coefficient is zero for all $j \leq k$. Let $p(x)$ be the polynomial $x(x-1)(x-2)\cdots(x-j+2)$. This is a polynomial of degree less than k . The sum on the right for the j th coefficient becomes

$$\sum_{i=0}^k (-1)^i \binom{k}{i} \frac{p(i)}{(j-1)!}.$$

In order to show that this is zero, consider the binomial expansion of $(1+x)^k$:

$$(1+x)^k = \sum_{i=0}^k \binom{k}{i} x^i.$$

Taking the derivative with respect to x $(j-1)$ -times on both sides yields

$$k \cdot (k-1) \cdots (k-j+2) (1+x)^{k-j+1} = \sum_{i=0}^k \binom{k}{i} i(i-1) \cdots (i-j+2) x^{i-j+1},$$

where we have $p(i)$ on the right side multiplying x^{i-j+1} . Setting $x = -1$ gives

$$0 = \sum_{i=0}^k (-1)^{i-j+1} \binom{k}{i} p(i).$$

Then we just multiply by $(-1)^{j-1}/(j-1)!$ to obtain the lemma. \square

3.2. Labeling edges in H_A^n .

By the remark immediately following Theorem 2.6, $X_{\mathbb{H}}$ contributes to the first $n-1$ coefficients of $\langle D^n \rangle$ if and only if $v(\mathbb{H}) - k(\mathbb{H}) \leq n-2$. Therefore, we may restrict to this set of spanning sub-graphs and show that

$$d^* \left(\sum_{v(\mathbb{H}) - k(\mathbb{H}) \leq n-2} X_{\mathbb{H}} \right) \leq M(D^n) - 4(n-1),$$

to prove Theorem 2.4. Our strategy is to divide up the sum over equivalence classes where Lemma 3.1 can be repeatedly applied. In order to define this equivalence relation, we introduce labels and define sequences for each \mathbb{H} that satisfies $v(\mathbb{H}) - k(\mathbb{H}) \leq n-2$ in this section.

Since D is not A -adequate, there is an edge in the all- A state graph of D with two ends on the same state circle S , such that it lies in a region bounded by S on the projection 2-sphere. We name the crossing in D corresponding to this edge by e , and denote the n^2 crossings corresponding to the cable of e in D^n by e^n . The set of crossings in e^n corresponding to n loops in the all- A state graph H_A^n will be denoted by e_ℓ^n .

See Figure 3 for the local picture of the resulting state graph where we choose the A -resolution at all the crossings in e^n .

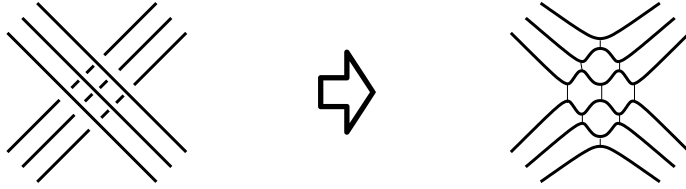


FIGURE 3. Identical local picture of the A -resolution of the 3-cable of a crossing.

Recall that H_A^n is the all- A state graph of D^n associated to the spanning subgraph of \mathbb{G}_A^n with no edges. We detail below labeling conventions for the edges corresponding to e^n in H_A^n , and we will just refer to these edges by e^n . For an illustration of the labels, where we locally straighten the strands, see Figure 4.

- (a) In H_A^n there will be n state circles corresponding to the cabling of S in D^n . We denote the state circle where the two ends of edges in e_ℓ^n are by S_0 . Then, S_i is the state circle where one of its complementary regions on S^2 contains only e_ℓ^n and S_0, \dots, S_{i-1} . The state circles S_0, S_1, \dots, S_{n-1} divide the sphere into $n+1$ regions. We denote the region with boundary S_{i-1} and S_i for $i = 1, 2, \dots, n-1$ by Ω_i and the region with boundary S_0 containing e_ℓ^n by Ω_0 . With this labeling we can refer to edges in e^n by which regions they are in. See Figure 4(a).
- (b) For each i , an edge in Ω_i corresponding to a crossing in e^n will be labeled by odd or even integers. First we label the edges in e_ℓ^n by consecutive odd integers $1, 3, 5, \dots, 2n-1$. For $i \neq 0$, an edge in Ω_i with an end on S_{i-1} and an end on S_i is labeled with the integer m if the end of the edge on S_{i-1} is between the ends of two edges labeled $m-1$ and $m+1$. See Figure 4(b).
- (c) Drawing a curve through all the edges in Ω_0 and dividing the projection sphere in half, an edge of e^n in Ω_i for $i \geq 1$ will be labeled L or R depending on whether it is on one side of the curve or the other side, respectively, so $L\Omega_i$ and $R\Omega_i$ refer to all the edges in e^n in region Ω_i on the left side or on the right side, and we just let $L\Omega_0 = R\Omega_0 = \Omega_0$. This notation, along with the numbering of edges of e^n , specifies each edge in e^n uniquely. See Figure 4(c), the dividing curve is shown in dotted red.

The crossings in e^n inherit the labels from the corresponding edges in H_A^n , so that for any state graph of D^n , a segment in the state graph resulting from choosing

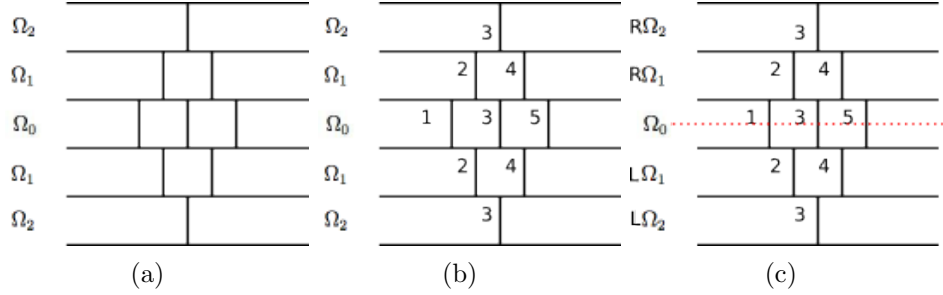


FIGURE 4. From left to right: an illustration of the labeling on the edges of e^n when $n = 3$.

a resolution at a crossing of e^n will also have the same label as the crossing. The numbers labeling e^n induce an ordering as well. We say that an edge with label a is *smaller* (resp. *greater*) than an edge with label b if $a < b$ (resp. $b > a$). Two edges are *equal* if their labeling numbers are equal. An edge with label c is *between* edges with label a and b if $a < c < b$. As long as it is clear which region Ω_i we are referring to, we will refer to the edges by their labeling numbers and by which side they are on. For example, $La \in \Omega_i$ and $a \in L\Omega_i$ for a an integer both indicate the edge in $L\Omega_i$ with label a .

Each edge in H_A^n uniquely corresponds to an edge of \mathbb{G}_A^n . We say that an edge in H_A^n is *included* in a spanning sub-graph $\mathbb{H} \subseteq \mathbb{G}_A^n$, if \mathbb{H} includes the corresponding edge in \mathbb{G}_A^n . We represent \mathbb{H} by indicating which edges of H_A^n are included in \mathbb{H} . Schematically, the edges that are included will be shown in red on H_A^n . This representation will allow us to compare the number of state circles in H , the state graph associated to \mathbb{H} , to that of H_A^n . We do this by modifying the state circles $s_A(D^n)$ of H_A^n locally at each crossing which corresponds to an edge included in \mathbb{H} , replacing the A -resolution by the B -resolution. This will be crucial to proving the lemmas in this section. See Figure 5 and 6 for exactly how H_A^n is modified to a different state graph when one changes from the A -resolution to the B -resolution at a crossing.

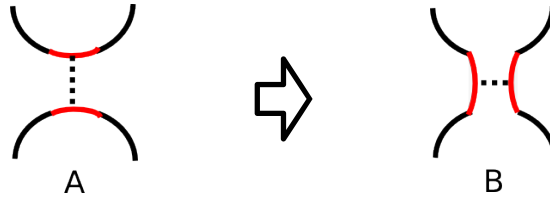


FIGURE 5. Locally, we replace the two segments making up the state circle (in red) and the segment recording the crossing location (dotted) of the A -resolution by that of the segments coming from the B -resolution.

Recall that our goal is to organize the spanning sub-graphs \mathbb{H} with $v(\mathbb{H}) - k(\mathbb{H}) \leq n - 2$ into equivalence classes where Lemma 3.1 can be applied. In [20], this was

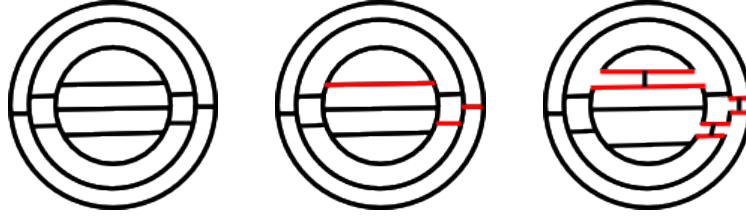


FIGURE 6. From left to right: an all A -state graph with 3 state circles; an edge inclusion representing a spanning sub-graph of the ribbon graph constructed from the all A -state graph; the state graph associated to the spanning sub-graph, which has 2 state circles.

done for spanning sub-graphs \mathbb{H} with $v(\mathbb{H}) - k(\mathbb{H}) \leq 1$. Two spanning sub-graphs are in the same equivalence class if they only differ by what they include from e_ℓ^n . The inclusion of each edge from e_ℓ^n increases the number of faces by one, since e_ℓ^n are one-edge loops for all of their associated state graphs. As a result, we set up sums of the form

$$\sum_{i=0}^{e_\ell^n} A^{e(\mathbb{G}_A^n) - 2i} (-A^2 - A^{-2})^{f(\mathbb{H}_0) + i - 1},$$

where \mathbb{H}_0 is the member of the equivalence class that does not include any edges from e_ℓ^n , and Lemma 3.1 can then be applied.

When $v(\mathbb{H}) - k(\mathbb{H}) > 1$, the situation is more complicated. We can no longer set up equivalence classes based on edge inclusions in e_ℓ^n , because it is no longer true that inclusion of an edge in e_ℓ^n increases the number of faces by one. Instead, we generalize this specific example to defining a set of edges $G(\mathbb{H})$ for a spanning sub-graph \mathbb{H} , where the increase in the number of faces is the same as the number of edges included. In order to control the change in the number of faces from edge inclusions, we make use of the explicit labelings just introduced, and we use Lemma 2.8 to compute $f(\mathbb{H})$ from the state graph H .

We first define $a(\mathbb{H})$, which determines the set of edges in e_ℓ^n we can add to \mathbb{H} in any combination while splitting off as many state circles as the edges added.

Definition 3.2. Given a spanning sub-graph $\mathbb{H} \subset \mathbb{G}_A^n$, we define \mathbb{H}^ℓ to be the spanning sub-graph obtained from \mathbb{H} by deleting from \mathbb{H} all the edges outside of the region Ω_1 . Consider the associated state graph H^ℓ . Let $m \in e_\ell^n$ be the smallest edge that has two ends on distinct state circles in H^ℓ , and $M \in e_\ell^n$ be the largest. We define $a(\mathbb{H})$ to be the subset of edges $c \in e_\ell^n$ such that $m \leq c \leq M$, and the empty set if every edge in e_ℓ^n has two ends on the same state circle in H^ℓ . For example, $a(\mathbb{H})$ is empty for the spanning sub-graph with no edges and sub-graphs for which only a single edge in Ω_1 is included.

If $a(\mathbb{H})$ is empty, then we can group them with other spanning sub-graphs where the only difference is which edges they include from e_ℓ^n and apply Lemma 3.1. For spanning sub-graphs where $a(\mathbb{H})$ is non-empty, we define sequences which controls

the quantity $v(\mathbb{H}) - k(\mathbb{H}) + g(\mathbb{H})$. The following lemma allows us to define the first terms of the sequences.

Lemma 3.3. *Suppose $a(\mathbb{H})$ is not empty and $v(\mathbb{H}) - k(\mathbb{H}) \leq n - 2$. Let m, M be the smallest and largest edge in $a(\mathbb{H})$, respectively.*

- (a) *If $1 \notin a(\mathbb{H})$, at least one of the pair of edges $R(m - 1), L(m - 1)$ in Ω_1 is included in \mathbb{H} . If $2n - 1 \notin a(\mathbb{H})$, at least one of the pair of edges $R(M + 1), L(M + 1)$ is included in \mathbb{H} . If $2n - 1 \in a(\mathbb{H})$, then an edge in Ω_1 with an end attached to the portion of Ω_0 cut off by $2n - 1$ is included in \mathbb{H} . See Figure 7 for an illustration.*
- (b) *If $1 \in a(\mathbb{H})$: Let \mathbb{H}^e be the sub-graph obtained from \mathbb{H} by deleting all the edges in e^n and in the region Ω_{n-1} . There is an integer $0 \leq k \leq n - 2$ such that the edge with the smallest labeling number in $R\Omega_k$ has two ends on the same state circle in the state graph H^e associated to \mathbb{H}^e .*

Proof.

- (a) The proof for the cases $1 \notin a(\mathbb{H})$, $2n - 1 \notin a(\mathbb{H})$, or $2n - 1 \in a(\mathbb{H})$ are similar, and we treat them together here.

This follows from the fact that the edges m and M need to be attached to separate state circles in H^ℓ , which has a simple picture, see Figure 7, only involving modifying the state circles S_0, S_1 of H_A^n based on edge inclusions of \mathbb{H} in Ω_1 . Requisite edges of Ω_1 need to be included in \mathbb{H} for m, M to have two ends on distinct state circles in the associated state graph.

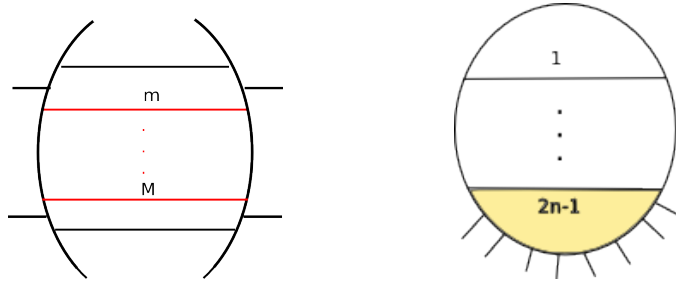


FIGURE 7. On the left: At least one of the two edges smaller than m in Ω_1 needs to be included in order that m ends up with two ends on distinct state circles in H^ℓ . The same is true with the edges bigger than M . On the right, the portion of the region Ω_0 cut off by the edge $2n - 1$ is shown in yellow, with the edges with one end attached to it.

- (b) For the case $1 \in a(\mathbb{H})$.

We show that there exists an index $0 \leq i \leq n - 2$ such that the edge with label $i + 1$ in $R\Omega_i$ has two ends on the same state circle in H^e . For each edge labeled $i + 1$ in $R\Omega_i$, we let the pieces of state circles where the two ends of the edge are on in H^e inherit the same labeling as in H_A^n , so the top and the bottom pieces will be denoted as S_i and S_{i-1} , respectively. If the

two boundaries are joined to each other in H^e , then the edge $i + 1$ will have two ends on the same state circle in H^e .

We show the following statement by induction on i , assuming that $a(\mathbb{H})$ is non-empty and contains the edge labeled 1.

- Let $0 < i \leq n - 2$. If for all $0 \leq r \leq i$, the edge labeled r in $R\Omega_r$ has two ends on distinct state circles in H_e , then two edges $e_i \in \Omega_i$ and $e_{i+1} \in \Omega_{i+1}$ that are not in e^n , are included in \mathbb{H}^e . Moreover, when we traverse the piece of the state circle S_i in H_A^n in the clockwise direction, we meet an end of e_i before we meet an end of e_{i+1} , and no edges between e_i and e_{i+1} in Ω_{i+1} are included in \mathbb{H}^e .

This statement implies the lemma for the case $1 \in a(\mathbb{H})$ since, if the k as in the lemma does not exist, then the index $n - 3$ satisfies the criteria for the statement. We then have that there is an edge e_{n-3} and e_{n-2} as in the statement, but in \mathbb{H}_e no edge is included in Ω_{n-1} . In $R\Omega_{n-2}$, the edge with label $n - 2$ will have two ends on the same state circle since S_{n-3} would have nowhere to exit except to join back with S_{n-2} . This is a contradiction.

For $i = 1$, if the edge labeled 1 in Ω_0 has two ends on distinct state circles in H^ℓ , an edge $e_1 \in \Omega_1$ that is not in e^n must be included in \mathbb{H} . Otherwise, the boundaries on S_0 where the two ends of edge 1 lie will join together and become the boundary of a single state circle in H^ℓ . If now the edge labeled 2 in $R\Omega_1$ also has two ends on distinct state circles in \mathbb{H}^e , we show that an edge $e_2 \in R\Omega_2$ fitting the description of the statement must also be included in \mathbb{H}^e . Since applying a Kauffman state to obtain a state graph gives disjoint state circles, the boundary of S_1 in H^e can only exit the region enclosed by the state circle consisting of S_0 by having \mathbb{H}^e include an edge e'_2 in Ω_2 . If we traverse S_1 in the clockwise direction we would meet an end of e_1 before we meet an end of e'_2 . Take the closest e'_2 to e_1 in the counter-clockwise direction to be e_2 . This proves the base case. See Figure 8 for an illustration where we straighten the strands locally.

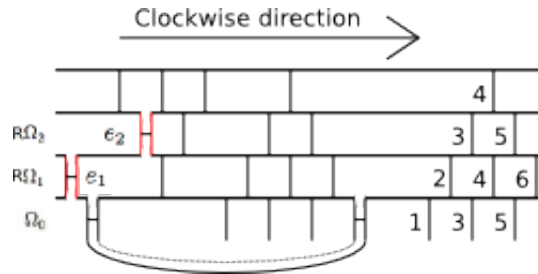


FIGURE 8. The base case $i = 1$.

Now for the induction step: Assume that the statement is true for $i - 1$. We already have edges e_1, \dots, e_i , each successive e_r, e_{r+1} fitting the description of the statement by the induction hypothesis. So we only need to show that e_{i+1} as described in the statement exists. The state circle with boundary S_{i-1} will enclose a region in H^e where the boundary S_i must exit

through an edge $e_{i+1} \in \Omega_{i+1}$ to avoid joining with S_{i+1} . Thus, an edge in $R\Omega_{i+1}$ must be included in H^e . Again we pick the one furtherest in the counter-clockwise direction to be e_{i+1} . This completes the proof of the statement. \square

Definition 3.4. For each \mathbb{H} where $a(\mathbb{H})$ is non-empty, we define s_a , an edge in H_A^n , according to the following rule:

- If $a(\mathbb{H})$ contains 1: Pick the smallest index $k \geq 0$ for which the edge labeled $k + 1$ in $R\Omega_k$ has two edge on the same state circle in H_e . This exists by Lemma 3.3. Let s_a be the edge labeled $k + 1$ in $R\Omega_k$ if the edge labeled k in $R\Omega_{k-1}$ is not included in \mathbb{H} , and let s_a be the edge labeled k in $R\Omega_{k-1}$ otherwise.
- If $a(\mathbb{H})$ does not contain 1: Let m be the edge with the smallest label in $a(\mathbb{H})$, we define

$$s_a := \begin{cases} R(m-1) \text{ in } \Omega_1, & \text{if } R(m-1) \text{ in } \Omega_1 \text{ is included in } \mathbb{H}. \\ L(m-1) \text{ in } \Omega_1, & \text{if } R(m-1) \text{ in } \Omega_1 \text{ is not included in } \mathbb{H}. \end{cases}$$

The edge s_a provides us with a starting point to find an edge with two ends on the same state circle in the state graph associated to \mathbb{H} , that is not included in \mathbb{H} originally. If such an edge is included in \mathbb{H} , the spanning sub-graph \mathbb{H}' resulting from this inclusion will have $f(\mathbb{H}') = f(\mathbb{H}) + 1$, and $v(\mathbb{H}') - k(\mathbb{H}') + g(\mathbb{H}') = v(\mathbb{H}) - k(\mathbb{H}) + g(\mathbb{H})$ as desired. The sequences $\{t^j\}$, $\{b^j\}$ to be defined below will allow us to keep track of any edge-inclusions of \mathbb{H} that may result in an edge that has ends on distinct state circles in the state graph associated to \mathbb{H} .

Definition 3.5. Consider a spanning sub-graph \mathbb{H} with s_a as in Definition 3.4. Let $i = 0, 1, 2, \dots, n-1$, we will define for each i and each side L, R , sequences of edges $\{t^j\}$ and $\{b^j\}$ in $L\Omega_i, R\Omega_i$ in H_A^n . Recall that m, M are the smallest and largest labels of the edges in $a(\mathbb{H})$. See Figure 9 for an example.

- If s_a is in $R\Omega_k$, we define the sequences $\{t^j\}$, $\{b^j\}$ for $R\Omega_i$ first.

$$t^0 = b^0 := \begin{cases} s_a & \text{if } i = k \\ \text{the smallest edge in } R\Omega_i \text{ greater than } b^0 \text{ of } R\Omega_{i-1} & \text{if } i > k \\ \text{the smallest edge in } R\Omega_i \text{ greater than } b^0 \text{ of } R\Omega_{i+1} & \text{if } i < k. \end{cases}$$

- We set the first term of the sequences of $L\Omega_0$ equal to the first term of the sequences of $R\Omega_0$. For $L\Omega_i$, the first terms of the sequences are given by

$$t^0 = b^0 := \text{the smallest edge in } L\Omega_i \text{ greater than } b^0 \text{ of } L\Omega_{i-1}.$$

- If s_a is in $L\Omega_k$, we replace R by L and L by R in the above definitions and we get the definition for the first terms of the sequences for this case.
- The rest of the sequences is given recursively by:

For $t^0 \in R\Omega_i$ (resp. $L\Omega_i$),
 $t^j :=$ the smallest edge $\leq M$ in $R\Omega_{i+j \pmod 2}$ (resp. $L\Omega_{i+(j \pmod 2)}$) included in \mathbb{H} greater than t^{j-1} . If there is no such edge then it is empty.

For $b^0 \in R\Omega_i$ (resp. $L\Omega_i$),
 $b^j :=$ the smallest edge $\leq M$ in $R\Omega_{i-j \pmod 2}$ (resp. $L\Omega_{i-j \pmod 2}$) included in \mathbb{H} greater than b^{j-1} . If there is no such edge then it is empty.

When $i = 0$, only the sequences $\{t^j\}$ are defined for $R\Omega_0$ and $L\Omega_0$ by the definitions above. One has terms from $R\Omega_1$ and another has terms from $L\Omega_1$, respectively. We let $\{b^j\}$ for $R\Omega_0$ be the same sequence as $\{t^j\}$ of $L\Omega_0$ and $\{b^j\}$ for $L\Omega_0$ be the same sequence as $\{t^j\}$ of $R\Omega_0$.

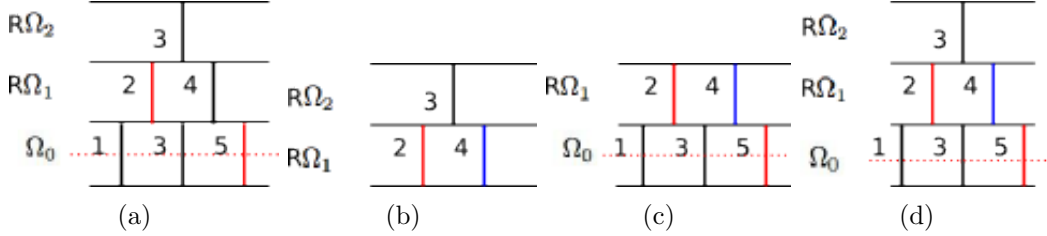


FIGURE 9. Let $s_a = 2 \in R\Omega_1$ for this example. The point of the sequences is to rule out edges in \mathbb{H} that may make edges after the edge labeled 2 in $R\Omega_1$ have two ends on distinct state circles in H . The sequence for $R\Omega_1$ are $\{b^j\} = \{2, 5, \emptyset, \dots\}$ and $\{t^j\} = \{2, \emptyset, \dots\}$.

Definition 3.6. Let the sequences $\{t^j\}$ and $\{b^j\}$ for $L\Omega_i$ or $R\Omega_i$ be given, and let $0 \leq i \leq n-2$. We define subsets t_i and b_i which we use to define $G(\mathbb{H})$, a subset of edges in H_A^n . Recall that an edge with label b is *between* edges labeled a and c if $a < b < c$. See Figure 10 for what kind of edges this picks out relative to the sequences.

For $t^0 \in R\Omega_i$ (resp. $L\Omega_i$),

- $t_i :=$ union of all edges of e^n in $R\Omega_i$ (resp. $L\Omega_i$) between t^j and t^{j+1} for j even and t^j non-empty. If t^{j+1} is empty then we take all edges after tR_i^j and $\leq M$ in $R\Omega_i$ (resp. $L\Omega_i$).
- $b_i :=$ union of all edges of e^n in $R\Omega_i$ (resp. $L\Omega_i$) between b^j and b^{j+1} for j even and b^j non-empty. If b^{j+1} is empty then we take all edges after bR_i^j and $\leq M$ in $R\Omega_i$ (resp. $L\Omega_i$).

Finally, we define

$$G(\mathbb{H}) := (\cup_{t^0 \in R\Omega_i} (t_i \cap b_i)) \cup (\cup_{t^0 \in L\Omega_i} (t_i \cap b_i)) \cup s_a$$

if $1 \in a(\mathbb{H})$ and

$$G(\mathbb{H}) := (\cup_{t^0 \in R\Omega_i} (t_i \cap b_i)) \cup (\cup_{t^0 \in L\Omega_i} (t_i \cap b_i))$$

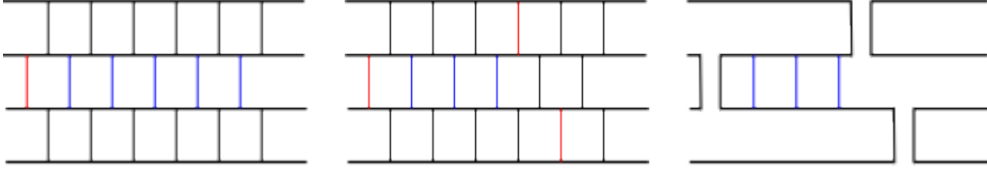


FIGURE 10. On the left: All the edges shown in blue here after the red edge are candidates to be included in $G(\mathbb{H})$, as long as there are no edges above and below after the middle red edge that are included in \mathbb{H} . In the middle and on the right: The inclusion of the top and bottom edges in red means that only three edges in blue after red will be included in $G(\mathbb{H})$. This is so that they will still have two ends on the same state circle as shown on the right when we only resolve the crossings corresponding to the red edges.

if $1 \notin a(\mathbb{H})$. Lastly, If $a(\mathbb{H}) = \{\emptyset\}$, then $G(\mathbb{H}) := \{\emptyset\}$.

3.3. Equivalence classes based on $G(\mathbb{H})$.

The point of the string of definitions and lemmas leading to the definition of $a(\mathbb{H})$ and $G(\mathbb{H})$ is that it gives us a way to understand how the number of faces changes for a new spanning sub-graph obtained by adding edges. We define an equivalence relation that will put spanning sub-graphs into equivalence classes where we can control the differences in the combinatorial quantities $e(\mathbb{H})$ and $f(\mathbb{H})$, directly related to $v(\mathbb{H}) - k(\mathbb{H}) + g(\mathbb{H})$, from one member to another.

Definition 3.7. We define an equivalence relation R on the set of spanning sub-graphs $\mathbb{H} \subseteq \mathbb{G}_A^n$ where $v(\mathbb{H}) - k(\mathbb{H}) \leq n - 2$. We say that

$$\mathbb{H} \cong_R \mathbb{H}'$$

if

- (1) $a(\mathbb{H}) = a(\mathbb{H}')$.
- (2) $G(\mathbb{H}) = G(\mathbb{H}')$, and \mathbb{H} only differs from \mathbb{H}' by what edge it includes from $G(\mathbb{H}) \cup (e_\ell^n \setminus a(\mathbb{H}))$. Recall that e_ℓ^n is the set of loops with both ends on S_0 in H_A^n .

This is an equivalence relation because $a(\mathbb{H})$ and $G(\mathbb{H})$ are well-defined on \mathbb{H} . Within an equivalence class C of R , the members of C only differ from each other by edge inclusion in $G(\mathbb{H}) \cup (e_\ell^n \setminus a(\mathbb{H}))$. Let \mathbb{H} be a member of an equivalence class C and consider the spanning sub-graph \mathbb{H}_0 obtained from \mathbb{H} by deleting all the edges in $G(\mathbb{H}) \cup (e_\ell^n \setminus a(\mathbb{H}))$. This spanning sub-graph is well-defined for C since $a(\mathbb{H})$ and $G(\mathbb{H})$ are the same across C .

Lemma 3.8. Let C be an equivalence class of the relation R , let \mathbb{H} be a member of C and \mathbb{H}_0 be the spanning sub-graph obtained from \mathbb{H} by deleting all the edges in $G(\mathbb{H}) \cup (e_\ell^n \setminus a(\mathbb{H}))$. Then

- (a) \mathbb{H}_0 is also in C .

- (b) *Adding any combination of k edges in $G(\mathbb{H}) \cup (e_\ell^n \setminus a(\mathbb{H}))$ to \mathbb{H}_0 gives another member \mathbb{H}' of C . In addition, $f(\mathbb{H}') = f(\mathbb{H}_0) + k$ and*

$$v(\mathbb{H}') - k(\mathbb{H}') + g(\mathbb{H}') = v(\mathbb{H}_0) - k(\mathbb{H}_0) + g(\mathbb{H}_0).$$

Proof. For (a), we need to show that $a(\mathbb{H}_0) = a(\mathbb{H})$ and $G(\mathbb{H}_0) = G(\mathbb{H})$. Consider the spanning sub-graph \mathbb{H}_0^ℓ obtained from \mathbb{H}_0 by deleting all the edges in Ω_0 and outside of Ω_1 as in Definition 3.2. Let m and M be the smallest edge and the largest edge in $a(\mathbb{H})$. The set $a(\mathbb{H}_0)$ would only differ from $a(\mathbb{H})$ if we delete edges in Ω_1 that are not between m and M . However, the algorithm for $G(\mathbb{H})$ requires that all edges in $G(\mathbb{H})$ are between m and M . Adding or deleting edges in $G(\mathbb{H}) \cup (e_\ell^n \setminus a(\mathbb{H}))$ does not affect \mathbb{H}^e , as used in Lemma 3.3 to define s_a , and the sequences, since $G(\mathbb{H}) \cup (e_\ell^n \setminus a(\mathbb{H})) \subset e^n$, so $G(\mathbb{H}_0) = G(\mathbb{H})$ and we have that \mathbb{H} and \mathbb{H}_0 are in the same equivalence class.

For (b), an edge is in $G(\mathbb{H}_0)$ if and only if it is between a pair of edges e and e' where $e \in R\Omega_i$ and $e' \in R\Omega_{i+1}$ or $e' \in R\Omega_{i-1}$, and there are no edges in $R\Omega_{i-1}$ or $R\Omega_{i+1}$ included between e and e' , or it is s_a . We have the same condition for an edge in $G(\mathbb{H}_0)$ on the left side by switching all the labels from R to L . See Figure 11 for an example.

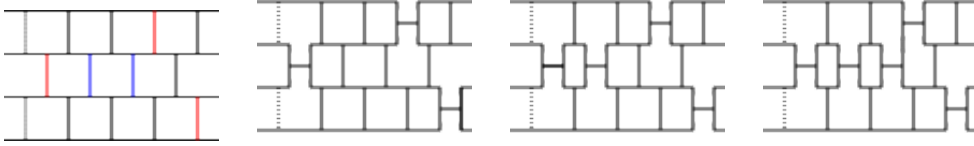


FIGURE 11. From left to right: An example of edges in $G(\mathbb{H})$ marked in blue; the state graph resulting from including zero edges from $G(\mathbb{H})$; the state graph resulting from choosing one edge from $G(\mathbb{H})$, note that a state circle splits off; the state graph resulting from including two edges from $G(\mathbb{H})$ with two state circles split off.

By design and since we are beginning the sequences at s_a which are especially chosen through using Lemma 3.3, all the edges in $G(\mathbb{H}_0) \cup (e_\ell^n \setminus a(\mathbb{H}))$ have two ends on the same state circle in state graph H_0 associated to \mathbb{H}_0 , with the identical local picture as shown in Figure 11: they all have two ends on the same state circle in H_0 , and none of the pairs of edges from $G(\mathbb{H}_0) \cup (e_\ell^n \setminus a(\mathbb{H}))$ lies embedded on distinct sides of a state circle such that only one end of an edge lies between the two ends of another. Thus, including any combination of k edges to \mathbb{H}_0 splits off k additional state circles in the new spanning sub-graph \mathbb{H} , and this shows the first part of (b).

Since we cannot add an edge in region Ω_i without there already being an edge included in Ω_i from the definition of the sequences which define $G(\mathbb{H})$, we also have that $v(\mathbb{H}') - k(\mathbb{H}') = v(\mathbb{H}_0) - k(\mathbb{H}_0)$. This also shows that $g(\mathbb{H}') = g(\mathbb{H}_0)$ since the increase in the number of faces is the same as the number of edges included. \square

3.4. Combinatorics of $G(\mathbb{H})$ and ribbon graphs.

By Lemma 3.8, if C is an equivalence class of the relation R , and $r = |G(\mathbb{H}) \cup (e_\ell^n \setminus a(\mathbb{H}))|$ is the cardinality of C , we may write

$$\sum_{\mathbb{H} \in C} X_{\mathbb{H}} = \sum_{i=0}^r \binom{r}{i} A^{e(\mathbb{G}_A^n) - 2e(\mathbb{H}_0) - 2r} (-A^2 - A^{-2})^{f(\mathbb{H}_0) + r - 1}.$$

For each equivalence class C , we are almost ready to use Lemma 3.1 to show the cancellation of the top coefficients of this sum. However, we need an estimate on r so that we know how many coefficients are cancelled. We also need to know how the resulting upper bound on the maximum degree compares with $M(D^n)$. Our main result of this section is the following:

Lemma 3.9. *If a spanning sub-graph $\mathbb{H} \subset \mathbb{G}_A^n$ satisfies $v(\mathbb{H}) - k(\mathbb{H}) + g(\mathbb{H}) \leq n - 2$, then*

$$|G(\mathbb{H})| + v(\mathbb{H}) - k(\mathbb{H}) + g(\mathbb{H}) \geq |a(\mathbb{H})| - 1.$$

This lemma connects the cardinality of $G(\mathbb{H})$ to the combinatorial ribbon graph quantities $v(\mathbb{H})$, $k(\mathbb{H})$, and $g(\mathbb{H})$ and is the last important piece of the proof of Theorem 2.4. The reader may skip ahead to Section 3.5 to see how it is applied and come back to this section later. To prove this lemma, we make use of the explicit labeling from Section 3.2.

Lemma 3.10. *Let e_1 be an edge in $R\Omega_i$ included in a spanning sub-graph \mathbb{H} . Suppose another spanning sub-graph \mathbb{H}' only differs from \mathbb{H} by including two additional edges e_2 in $R\Omega_i$ and e_3 in $R\Omega_{i+1}$ that satisfy the following criteria:*

- The edges e_2 and e_3 are not included in \mathbb{H} .
- The edge e_2 is between e_1 and e_3 .
- None of the edges in $R\Omega_i$ and $R\Omega_{i-1}$ between e_1 and e_3 are included in \mathbb{H} , and none of the edges in $R\Omega_{i+1}$ between e_1 and e_2 and between e_2 and e_3 are included in \mathbb{H} .

Then $v(\mathbb{H}') - k(\mathbb{H}') + g(\mathbb{H}') = v(\mathbb{H}) - k(\mathbb{H}) + g(\mathbb{H}) + 1$. The same statement for $L\Omega_i$ can be obtained by replacing all the labels from R to L .



FIGURE 12. On the left we have \mathbb{H} and on the right we have \mathbb{H}' . Their local pictures only differ in the inclusion of two edges e_1 and e_2 .

Proof. An example satisfying the hypotheses of the lemma is shown in Figure 12. By the definition of $g(\mathbb{H})$,

$$\begin{aligned} v(\mathbb{H}) - k(\mathbb{H}) + g(\mathbb{H}) &= \frac{2v(\mathbb{H}) - 2k(\mathbb{H}) + 2k(\mathbb{H}) - v(\mathbb{H}) + e(\mathbb{H}) - f(\mathbb{H})}{2} \\ &= \frac{v(\mathbb{H}) + e(\mathbb{H}) - f(\mathbb{H})}{2}. \end{aligned}$$

Compare the state graph H and H' . Including the edge e_3 will first split off a state circle. Then, including e_2 joins the state circle split off by e_1, e_3 to another state circle. The end result is that the number of states circles of \mathbb{H} and \mathbb{H}' are the same. By Lemma 2.8, this means $f(\mathbb{H}') = f(\mathbb{H})$. Therefore,

$$\begin{aligned} v(\mathbb{H}') - k(\mathbb{H}') + g(\mathbb{H}') &= \frac{v(\mathbb{H}') + e(\mathbb{H}') - f(\mathbb{H}')}{2} \\ &= \frac{v(\mathbb{H}) + e(\mathbb{H}) + 2 - f(\mathbb{H})}{2} \\ &= v(\mathbb{H}) - k(\mathbb{H}) + g(\mathbb{H}) + 1. \end{aligned}$$

□

We define $g_a(\mathbb{H})$, which counts the occurrences of edge inclusions described in Lemma 3.10 from the sequences $\{t^j\}$, $\{b^j\}$ in Definition 3.5.

Definition 3.11. If s_a as in Definition 3.4 is in $R\Omega_k$, we define $g_a(\mathbb{H})$ by taking the union of tg_i and bg_i defined for $L\Omega_i$ and $R\Omega_i$ below:

For $R\Omega_i$ (resp. $L\Omega_i$) and the sequences $\{t^j\}$, $\{b^j\}$.

- For all i , tg_i is the number of even ℓ 's such that t^ℓ is non-empty and between two non-empty terms b^j and b^{j+1} where j is even.
- For $i \neq k$, bg_i is the number of even ℓ 's such that b^ℓ is non-empty and between two non-empty terms t^j and t^{j+1} where j is even.
- bg_k is $\left\lfloor \frac{\#\{b^j\}}{2} \right\rfloor$ where $\#\{b^j\}$ is the number of non-empty terms in the sequence $\{b^j\}$, and $\lfloor \cdot \rfloor$ is the floor function which rounds down to the nearest integer.

Lastly,

$$g_a(\mathbb{H}) := \sum_{0 \leq i \leq k \text{ for each } R\Omega_i} bg_i + \sum_{k \leq i \leq n-2 \text{ for each } R\Omega_i} tg_i + \sum_{1 \leq i \leq n-2 \text{ for each } L\Omega_i} tg_i.$$

If s_a is on the left side, we replace L by R and R by L in the above definition to get the definition of $g_a(\mathbb{H})$ in this case.

By Lemma 3.10, $g_a(\mathbb{H})$ counts the quantity $v(\mathbb{H}) - k(\mathbb{H}) + g(\mathbb{H})$ locally. Now we are ready to show Lemma 3.9.

Proof. We assume first that $s_a \in R\Omega_k$ and M is the largest label of the edges in $a(\mathbb{H})$. The case where $s_a \in L\Omega_k$ is the same by reflection. Recall that $G(\mathbb{H})$ is defined in Definition 3.6 by taking the union of the intersection $b_i \cap t_i$ for all i

ranging over $R\Omega_i$ and $L\Omega_i$. The cardinality of the intersection $b_i \cap t_i$ is related to tg_i , bg_i , b_{i+1} , and t_{i-1} when $i \leq n-2$ for $R\Omega_i$ (resp. $L\Omega_i$) as follows.

$$(1) \quad \begin{aligned} |b_i \cap t_i| &= |b_i| - |tg_i| - |b'_i| \\ |b_i \cap t_i| &= |t_i| - |bg_i| - |t'_i|, \end{aligned}$$

where b'_i consists of edges in $R\Omega_i$ between b^j and b^{j+1} in the sequence $\{b^j\}$ for $R\Omega_{i+1}$ where j is even and b^j is non-empty. Similarly, t'_i consists of edges in $R\Omega_i$ between t^j and t^{j+1} in the sequence $\{t^j\}$ for $R\Omega_{i-1}$ where j is even and t^j is non-empty. These are the edges excluded from b_i and t_i by the intersection $b_i \cap t_i$, respectively.

We also have

$$(2) \quad \begin{aligned} |b_{i+1}| &\geq |b'_i| \text{ or } |b_{i+1}| = |b'_i| - 1. \\ |t_{i-1}| &\geq |t'_i| \text{ or } |t_{i-1}| = |t'_i| - 1. \end{aligned}$$

By Definition 3.6, an edge $\leq M$ of $R\Omega_i$ (resp. $L\Omega_i$) with sequence $\{b^j\}$ and $\{t^j\}$ is in b_i , if and only if it is between b^j and b^{j+1} where j is even and b^j is non-empty. An edge is excluded from b_i by the intersection $b_i \cap t_i$ if and only if it is between t^j and t^{j+1} where j is odd and b^j is non-empty. The sequence $\{b^j\}$ for $R\Omega_{i+1}$ is the sequence $\{t^j\}$ for $R\Omega_i$ with the first term removed. Therefore, b_{i+1} contains edges excluded from b_i with one added to all the labels. Since $R\Omega_{i+1}$ has one less edge than $R\Omega_i$, we have the two possibilities for $|b_{i+1}|$ and $|b_i|$. Similarly, we have the statement for $|t_i|$ and $|t_{i-1}|$ and for $L\Omega_i$.

We define the following quantities.

- $C := \#\{i : k \leq i < n-2, |b_{i+1}| = |b'_i| - 1\}$ for each $R\Omega_i$ such that the edge lost from b'_i is an edge coming from bR_{k+1} in the following sense: the edge $-r$ in $R\Omega_r$ is in b_r for each $k+1 \leq r \leq i$.
- $C' := \#\{i : 0 \leq i < n-2, |b_{i+1}| = |b'_i| - 1\}$, for each $L\Omega_i$ such that the edge lost from b'_i is an edge coming from tR_{k-1} in the following sense: the edge $-r$ is in b_r for each $0 \leq r \leq i$ of $L\Omega_r$ and in t_r for each $0 \leq r \leq k-1$ of $R\Omega_r$.
- $E := \#\{i : k < i < n-2, i \text{ is counted by } C' \text{ and } i+k \text{ is counted by } C\}$.

As discussed, an edge in b_{i+1} is an edge in b'_i with one added to the label, so if $|b_{i+1}| = |b'_i| - 1$ is counted by C , an edge with label $n-i$ is included in b_i . Similarly, an edge with label $n-i-1$ is included in b_{i-1} , and so on. We see that an edge with label $R(n-i-(i-k-1)) = R(n-2i+k+1)$ is included in b_{k+1} . Similarly, an index i counted by C' corresponds to an edge with label $R(n-(i-k)-(k+i-k-1)) = R(n-2i+k+1)$ included in t_{k-1} . Therefore, E counts the number of edges with the same labeling numbers in bR_{k+1} and tR_{k-1} .

To complete our estimate of $|G(\mathbb{H})|$, we show

$$(3) \quad |b_k \cap t_k| + |b_{k+1}| + |t_{k-1}| \geq |s_a| - |tg_k| - |bg_k| + \delta_k + E,$$

where $|s_a|$ is defined to be the number of edges greater than s_a and $\leq M$ in $R\Omega_k$, and δ_k is a quantity counted by C . To begin with, we have $|b_k| = |s_a| - |bg_k| - |t_{k-1}|$ since edges of the sequence $\{b^j\}$ for $R\Omega_k$ counted by bg_k are not included in b_k , and there are additional edges excluded from $|s_a|$ that are replaced by edges in t_{k-1} .

Then, $|b_k \cap t_k| + |b_{k+1}| \geq |s_a| - |bg_k| - |t_{k-1}| - |tg_k| - \delta_k + E$, where $\delta_k = 1$ if $|b_{k+1}| = |b'_k| - 1$ and 0 otherwise. So, δ_k is counted by C . The edges greater than s_a that are part of the sequence $\{t^j\}$ of $R\Omega_{k+1}$, counted by tg_k , will not be included in $b_k \cap t_k$. In addition, extra edges with the same labels as those in t_{k-1} may be included in b_{k+1} . These are counted by E . Adding $|t_{k-1}|$ to the left hand side of the inequality produces the inequality (3).

Putting the statements (1) and (2) together for each $0 \leq i \leq n-2$ and using the definition of $G(\mathbb{H})$, we have

$$\begin{aligned}
 |G(\mathbb{H})| &\geq |b_k \cap t_k| + \sum_{i \neq k \text{ for each } R\Omega_i} |b_i \cap t_i| + \sum_{i \neq 0 \text{ for each } L\Omega_i} |b_i \cap t_i| \\
 &\geq |b_k \cap t_k| + |b_{k+1}| + |t_{k-1}| \\
 (4) \quad &- C - C' - \sum_{0 \leq i < k \text{ for each } R\Omega_i} bg_i - \sum_{k < i \leq n-2 \text{ for each } R\Omega_i} tg_i - \sum_{1 \leq i \leq n-2 \text{ for each } L\Omega_i} tg_i.
 \end{aligned}$$

The inequality (3) then gives

$$(5) \quad |G(\mathbb{H})| \geq |s_a| - g_a(\mathbb{H}) + E - (C + C').$$

Consider the set of edges which are the last edges of the sequences $\{b_j\}$ for $R\Omega_i$ for all $k \leq i \leq n-1$. We define C'' to be the number of edges that are part of the decreasing portion of the set, and do not belong to sets counted by $g_a(\mathbb{H})$. In other words, an edge in $R\Omega_i$ is counted by C'' if there is an edge in the set in $R\Omega_{i-1}$ which is larger.

To link the combinatorial estimate of $|G(\mathbb{H})|$ to the quantity $v(\mathbb{H}) - k(\mathbb{H}) + g(\mathbb{H})$, we show

$$(6) \quad v(\mathbb{H}) - k(\mathbb{H}) + g(\mathbb{H}) \geq k - 1 + g_a(\mathbb{H}) + (C + C' - E) - C'',$$

and

$$(7) \quad |G(\mathbb{H})| \geq |s_a| - g_a(\mathbb{H}) + E - (C + C') + C''.$$

If \mathbb{H}' is obtained from \mathbb{H} by adding edges, then $v(\mathbb{H}') - k(\mathbb{H}') + g(\mathbb{H}') \geq v(\mathbb{H}) - k(\mathbb{H}) + g(\mathbb{H})$ because $f(\mathbb{H}') - f(\mathbb{H}) \leq$ the number of edges added [22, Lemma 5.6]. This shows that we can compute a lower bound of $v(\mathbb{H}) - k(\mathbb{H}) + g(\mathbb{H})$ for any spanning sub-graph \mathbb{H} by starting with the spanning sub-graph of no edges, \mathbb{H}_A^n , add a subset of edges of \mathbb{H} to it to obtain \mathbb{H}' for which we can compute $v(\mathbb{H}') - k(\mathbb{H}') + g(\mathbb{H}')$, then $v(\mathbb{H}') - k(\mathbb{H}') + g(\mathbb{H}')$ will be a lower bound for $v(\mathbb{H}) - k(\mathbb{H}) + g(\mathbb{H})$ since adding any additional edges will not decrease $v(\mathbb{H}) - k(\mathbb{H}) + g(\mathbb{H})$.

Recall that \mathbb{H}_A^n is the spanning sub-graph of no edges. We will be adding to \mathbb{H}_A^n two subsets of edges of \mathbb{H} :

- (a) By definition and the proof of Lemma 3.3, the edge s_a is in $R\Omega_k$ if for each $0 < i \leq k-1$, an edge in Ω_i that is not in e^n is included in \mathbb{H} . Denote this set of edges by V . The first set will be the union of V , the first terms of the sequences $\{b^j\}$ for $R\Omega_i$ where $0 \leq i \leq k$, and the largest term of the sequences $\{b^j\}$ and $\{t^j\}$, going through $L\Omega_i$ and $R\Omega_i$ for each i .
- (b) The second set are the edges in all the sequences $\{t^j\}$, $\{b^j\}$ defined for \mathbb{H} .

We denote the spanning sub-graph obtained by adding the set (a) to \mathbb{H}_A^n by \mathbb{H}^1 , and the spanning sub-graph obtained by adding the set (b) to \mathbb{H}^1 by \mathbb{H}^2 . The spanning sub-graph \mathbb{H} is obtained by adding the rest of the edges included in \mathbb{H} to \mathbb{H}^2 .

We first show that

$$(8) \quad v(\mathbb{H}^1) - k(\mathbb{H}^1) + g(\mathbb{H}^1) \geq k - 1 + (C + C' - E) - C''.$$

In \mathbb{H}^1 , the edges from V are edges between state circles S_i, S_{i+1} for $0 \leq i \leq k-2$ in H_A^n , the all- A state graph associated to \mathbb{H}_A^n , which increases as i increases. The edges counted by C are edges between state circles S_i, S_{i+1} for $k+1 \leq i \leq n-2$, and an edge in $R\Omega_i$ will be included in \mathbb{H}^1 if i is counted by C by construction. If C' is zero then it is clear that $v(\mathbb{H}^1) - k(\mathbb{H}^1) \geq k - 1 + C$. If C' is not zero then for each $0 < i < k$, an edge in $R\Omega_i$ and an edge in $R\Omega_{i+1}$ are included in \mathbb{H}^1 which increases in the counter-clockwise direction as i decreases. These are included in the last terms of the sequence $\{b^j\}$ for $R\Omega_i$. See Figure 13.

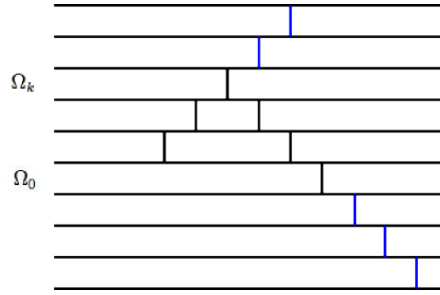


FIGURE 13. In this example, $k = 3$. A blue edge in $R\Omega_i$ for any i contributes 1 to the lower bound for $v(\mathbb{H}) - k(\mathbb{H}) + g(\mathbb{H})$. A blue edge in $L\Omega_i$ for $0 \leq i \leq k$ contributes 1 as well to the lower bound for $v(\mathbb{H}) - k(\mathbb{H}) + g(\mathbb{H})$.

Let i be an index where $L\Omega_i$ is counted by C' but not by E . Consider the corresponding index $i+k$ of $R\Omega_{i+k}$. Since $i+k$ is not counted by E , $R\Omega_{i+k}$ is not counted by C . Suppose that an edge in \mathbb{H}^1 is included in $R\Omega_{i+k}$. If it is not part of a decreasing sequence of edges, then we can increase our lower bound for $v(\mathbb{H}) - k(\mathbb{H})$ by 1 by counting this edge.

Let i' be the smallest index for which $L\Omega_{i'}$ is counted by C' but not by E , and there is no edge included in \mathbb{H} from $R\Omega_{i'+k}$. This means that there will be no more edges included from $R\Omega_r$ where $r \geq i' + k$ because the first term of the sequence $\{b^j\}$ for $R\Omega_r$ will be empty. It may be the case that for each index from i' to $i' + k$, $R\Omega_{i'}$ has already been counted by C . On the other hand, if there are indices $r \geq i' + k$ where $L\Omega_r$ is counted by C' , then each of those would increase the count for $v(\mathbb{H}^1) - k(\mathbb{H}^1)$ by one, since they would be the only edges connecting the state circles S_{r-1} and S_r . However, if $i' > k$, then there are k extra contributions to $g(\mathbb{H}^1)$ coming from adding edges outside of the edge inclusions from $R\Omega_1$ to $R\Omega_k$, see Figure 13, and this compensates for the indices from i' to $i' + k$ that may already be counted by C . This gives the inequality (8). This also takes care of the case where there are no more edges after $R\Omega_{i'}$ in an increasing sequence.

Adding the set (b) of edges from the rest of the sequences to get \mathbb{H}^2 , we applying Lemma 3.10 repeatedly. The increase in $v(\mathbb{H}^2) - k(\mathbb{H}^2) + g(\mathbb{H}^2)$ is given by $g_a(\mathbb{H})$ since we only counted the increase to $v(\mathbb{H}^1) - k(\mathbb{H}^1) + g(\mathbb{H}^1)$ from increasing sequence of edges in set (a). We get the inequality (6).

For (7), an edge in $R\Omega_i$ in a decreasing sequence corresponds to extra edges in b_{i+r} in $R\Omega_{i+r}$ where r is some integer > 0 and the last edge of $\{b_j\}$ for $R\Omega_{i+r}$ is part of an increasing sequence. This will either increase $G(\mathbb{H})$ or $g_a(\mathbb{H})$ independent of the quantities counted by C, C' . This shows (7), and we add the inequality (6) to (7). Since $|s_a| = a(\mathbb{H}) - 1$, we have Lemma 3.9. \square

3.5. Proof of Theorem 2.4.

To summarize, by Theorem 2.6, we have that

$$\langle D^n \rangle = \sum_{\mathbb{H} \subseteq \mathbb{G}_A^n} A^{e(\mathbb{G}_A) - 2e(\mathbb{H})} (-A^2 - A^{-2})^{f(\mathbb{H}) - 1}.$$

The contribution of a spanning sub-graph $\mathbb{H} \subset \mathbb{G}_A^n$ to $\langle D^n \rangle$ is given by

$$X_{\mathbb{H}} := A^{e(\mathbb{G}_A^n) - 2e(\mathbb{H})} (-A^2 - A^{-2})^{f(\mathbb{H}) - 1}.$$

Analysis of the monomials involved in the expansion (see [5], Theorem 6.1) show that a spanning sub-graph \mathbb{H} contributes to the first $n - 1$ coefficients if and only if $v(\mathbb{H}) - k(\mathbb{H}) + g(\mathbb{H}) \leq n - 2$. We organize the contributions of these spanning sub-graphs by putting them into equivalence classes defined by R as in Definition 3.7. Let C denote an equivalence class of R . Recall that d^* of a polynomial $p(A)$ in A is the maximum degree of $p(A)$. We have

$$d^* \langle D^n \rangle = d^* \sum_C X_C,$$

where

$$X_C := \sum_{\mathbb{H} \in C} X_{\mathbb{H}},$$

and C ranges over equivalence classes of R .

Let $r = |G(\mathbb{H}_0) \cup (e_\ell^n \setminus a(\mathbb{H}_0))|$, where \mathbb{H}_0 is the member in C which does not include any edges in $G(\mathbb{H}_0)$. By Lemma 3.9, $r \geq n - 1 - (v(\mathbb{H}_0) - k(\mathbb{H}_0) + g(\mathbb{H}_0))$. We also have, by Lemma 3.8,

$$X_C = \sum_{i=0}^r \binom{r}{i} A^{e(\mathbb{G}_A^n) - 2e(\mathbb{H}_0) - 2r} (-A^2 - A^{-2})^{f(\mathbb{H}_0) + r - 1}.$$

Applying Lemma 3.1 to this sum and writing $f(\mathbb{H}_0)$ in terms of $v(\mathbb{H}_0)$, $g(\mathbb{H}_0)$, and $k(\mathbb{H}_0)$ using the definition of genus,

$$\begin{aligned} d^* X_C &\leq c(D^n) - 2e(\mathbb{H}_0) + 2(-2g(\mathbb{H}_0) + 2k(\mathbb{H}_0) - v(\mathbb{H}_0) + e(\mathbb{H}_0) - 1) - 4r \\ &\leq c(D^n) - 2e(\mathbb{H}_0) + 2(-2g(\mathbb{H}_0) + 2k(\mathbb{H}_0) - v(\mathbb{H}_0) + e(\mathbb{H}_0) - 1) \\ &\quad - 4(n - 1 - (v(\mathbb{H}_0) - k(\mathbb{H}_0) + g(\mathbb{H}_0))) \\ &\leq c(D^n) + 2v(\mathbb{H}_0) - 2 - 4(n - 1) \\ &\leq n^2 c(D) + 2|s_A(D^n)| - 2 - 4(n - 1), \end{aligned}$$

where $n^2 c(D) + 2|s_A(D^n)| - 2 = M(D^n)$. Since $\deg X_C \leq M(D^n) - 4(n - 1)$ for all the equivalence classes C containing all spanning sub-graphs \mathbb{H} with $v(\mathbb{H}) - k(\mathbb{H}) \leq n - 2$, we see that $d^* \langle D^n \rangle \leq M(D^n) - 4(n - 1)$.

4. DETECTING SEMI-ADEQUACY USING THE COLORED JONES POLYNOMIAL

We use Theorem 1.1 to define a link invariant as in [20]. Let D be a diagram of an oriented link K . Let $c_-(D)$ be the number of negative crossings of D , and recall that $|s_A(D)|$ is the number of state circles in the all A -resolution of D , $c(D)$ is the number of crossings in D , and $w(D)$ is the writhe. We consider the complexity

$$(c_-(D), c(D), |s_A(D)| - w(D)),$$

ordered lexicographically. Let $\mathcal{D}(K)$ be the set of diagrams which minimizes this complexity.

Let $G_D(n, A)$ be as defined in Section 2.1. We let

$$M(G_D(n, A)) := M(D^{n-1}) + 4 - 2(n - 1) - w(D)((n - 1)^2 + 2(n - 1)).$$

Recall that a lower bound $h_n(D)$ (See Definition 2.3.) of the minimum degree $d(n)$ of $J_K(n, q)$ is $-\frac{1}{4}M(G_D(n, A))$.

Definition 4.1. Let K be a link and D an oriented link diagram in $\mathcal{D}(K)$. For $i \geq 1$, let $\beta_i = \beta_i(D)$ be the coefficient of $A^{M(G_D(i+2, A)) - 4(i-1)}$ in $G_D(i + 2, A)$. Define

$$J_D^A(q) := \sum_{i=1}^{\infty} \beta_i q^{i-1}.$$

We will need the following lemma from [20].

Lemma 4.2. [20, Lemma 3.4] *Suppose that for a link K , there is $D \in \mathcal{D}(K)$ that is A -adequate. Then, all the diagrams in $\mathcal{D}(K)$ are A -adequate.*

Applying Theorem 1.1, we have the following corollaries.

Corollary 4.3. $J_D^A(q) \neq 0$ if and only if D is A -adequate.

Proof. By definition, the coefficient β_i of $A^{M(G_D(i+2, A)) - 4(i-1)}$ in $G_D(i + 2, A)$ is the coefficient of $q^{h_{i+2}(D) + i - 1}$ in $J_K(i + 2, q)$. If D is not A -adequate, then Theorem 1.1 says that $d(i + 2) \geq h_{i+2}(D) + i$, so $\beta_i = 0$ for all i . This shows the forward direction. For the converse, β_1 is the coefficient of $q^{h_3(D)}$ in $J_K(3, q)$. If D is A -adequate, then

$h_3(D)$ is equal to the minimum degree $d(3)$ of $J_K(3, q)$, so $\beta_1 \neq 0$, and this shows that $J_D^A(q) \neq 0$. \square

Corollary 4.4. *The power series $J_D^A(q)$ defined above is independent of the diagram $D \in \mathcal{D}(K)$, thus it is an invariant of K , which we denote by $J_K^A(q)$.*

Proof. If K is not A -adequate, then any diagram in $\mathcal{D}(K)$ is not A -adequate. Let D be a diagram in $\mathcal{D}(K)$, then by Theorem 1.1, $J_D^A(q) = 0$. If K is A -adequate, then an A -adequate diagram of K minimizes the complexity $(c_-(D), c(D), |s_A(D) - w(D)|)$, thus it belongs to $\mathcal{D}(K)$, and all the diagrams in $\mathcal{D}(K)$ are A -adequate by Lemma 4.2. Let D be a diagram in $\mathcal{D}(K)$. As shown in [1], the colored Jones polynomial has a tail and $J_D^A(q)$ records the stable coefficients of the sequence $\{J_K(n, q)\}_{n=2}^\infty$, therefore it is also independent of the diagram D . \square

Definition 4.5. Consider the graph G_A with vertices the state circles, and edges the segments from the graph of the A -resolution H_A of D , we denote by $\chi_A(D)$ the Euler characteristic of G_A .

Corollary 4.6. *Suppose D is an A -adequate diagram of a link K and D' is another diagram of K . Then D' is A -adequate if and only if $c_-(D) = c_-(D')$ and $\chi_A(D) = \chi_A(D')$.*

Proof. If D' is A -adequate, then $c_-(D)$ and $|s_A(D)| - w(D)$ are invariants of K [22, Theorem 5.13]. Thus, $\chi_A(D) = |s_A(D)| - c(D) = |s_A(D)| - w(D) - 2c_-(D)$ is also an invariant of K . For the converse, since D is A -adequate, the minimum degree $d(n+1)$ of $J_K(n+1, q)$ is equal to $h_{n+1}(D)$, which we rewrite here slightly differently:

$$\begin{aligned} h_{n+1}(D) &= -\frac{1}{4}(n^2 c(D) + 2n|s_A(D)| - 2 + 4 - 2n - \omega(D)(n^2 + 2n)) \\ &= -\frac{1}{4}(2c_-(D)n^2 + 2n(|s_A(D)| - w(D)) + 2 - 2n) \end{aligned}$$

If D' is not A -adequate and $c_-(D) = c_-(D')$, $\chi_A(D) = \chi_A(D')$, then $s_A(D) - \omega(D) = s_A(D') - \omega(D')$, so $h_{n+1}(D) = h_{n+1}(D')$. Theorem 1.1 applied to D' will imply that $d(n+1) < h_{n+1}(D)$, which is a contradiction. \square

5. A WORKED OUT EXAMPLE

The proof of Theorem 2.4 uses the labeling on cables of a non A -adequate link diagram to separate the contributing spanning sub-graphs into equivalence classes. There is then cancellation of coefficients summing over the contributing polynomial of each member of the equivalence class. In this section, we illustrate this process on an example.

The link diagram shown on the left in Figure 14 is not A -adequate because there is an edge with both ends on the same state circle S in the all- A state graph of the diagram. This link may have an A -adequate diagram. However, our aim is to show the division of spanning sub-graphs into equivalence classes for this non A -adequate

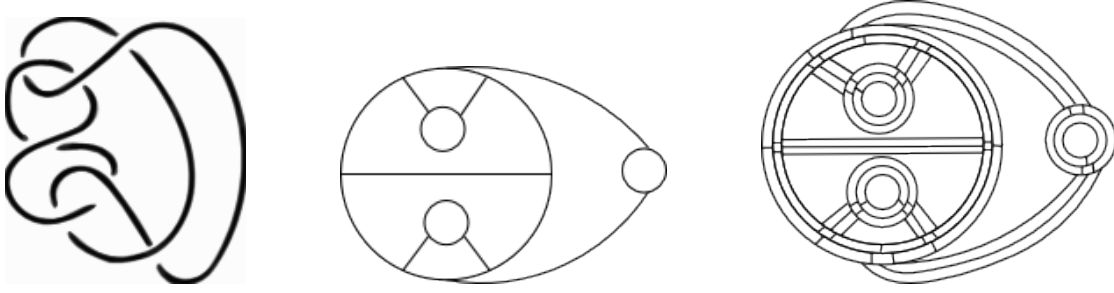


FIGURE 14. From left to right: a non A -adequate link diagram, the all- A state graph, and the all A -state graph of the 3-cable.

diagram. We consider the case $n = 3$ and apply the construction for the proof of Theorem 2.4. The theorem in this case says that $M(D^3) - d^*\langle D^3 \rangle \geq 4 \cdot (3 - 1)$.

The first step is to restrict to spanning sub-graphs $\mathbb{H} \subset \mathbb{G}_A^n$ with $v(\mathbb{H}) - k(\mathbb{H}) \leq 1$ by the remark following Theorem 2.6. Following the conventions of Section 3, e_ℓ^3 denotes the edges in the all A -state graph of D^3 with two ends on a state circle S_0 , S_1 is the state circle coming from cabling S in D^3 that contains S_0 and no other state circles coming from cabling S in one of its complementary regions on S^2 . The edges in the all- A state graph of D^3 corresponding to e_ℓ^3 have their labelings. We also label all the regions Ω_0 and Ω_1 . See Figure 15.

The restriction $v(\mathbb{H}) - k(\mathbb{H}) \leq 1$ means that only edges between a pair of distinct vertices, and one-edge loops e_ℓ^3 in \mathbb{G}_A^n , may be included in \mathbb{H} . There are two cases, one where $a(\mathbb{H})$ is empty and another where $a(\mathbb{H})$ is non-empty. The case where $a(\mathbb{H})$ is empty includes the case where the pair of distinct vertices with an edge included between them is not S_0 and S_1 , $v(\mathbb{H}) - k(\mathbb{H}) = 0$, and when a single edge or two edges with the same label in Ω_1 are included between S_0 and S_1 . The spanning sub-graph \mathbb{H} of each of these cases belongs to an equivalence class where the only difference between the members are which edges are included from e_ℓ^3 . Since $|e_\ell^3| = 3$, applying Lemma 3.1 will show that the contribution of \mathbb{H} to terms with power greater than $M(D^3) - 8$ in $\langle D^3 \rangle$ is equal to zero.

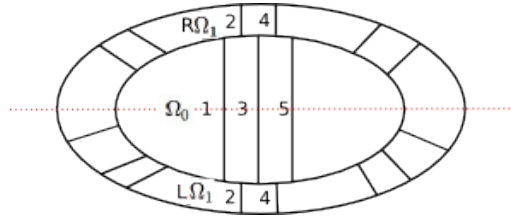


FIGURE 15.

We restrict to the local picture of \mathbb{G}_A^n involving only S_0 and S_1 . The case when $a(\mathbb{H})$ is non-empty is what motivates the technicalities in the proof of Theorem 2.4. We cannot put \mathbb{H} in an equivalence class where the only difference between the members is what edges it includes from e_ℓ^3 , since an edge in e_ℓ^3 may belong to $a(\mathbb{H})$ where it has two edges on distinct state circles in the state graph associated to

\mathbb{H} . Inclusion of the edge may then give a spanning sub-graph \mathbb{H}' with a different $V(\mathbb{H}') - k(\mathbb{H}') + g(\mathbb{H}')$, which determines the power of the term $(-A^2 - A^{-2})$ of the contribution $X_{\mathbb{H}}$. Lemma 3.1 will not apply to the sum of the contributions in this case.

On the other hand, if $a(\mathbb{H})$ is non-empty, then at least one edge must be included in \mathbb{H} between S_0 and S_1 and not between any other pairs of vertices which makes our restriction to the local picture valid. If only one edge between S_0 and S_1 is included then $a(\mathbb{H})$ is empty and we have already addressed this case. When more than one edge between S_0 and S_1 is included in \mathbb{H} , we look to the edges in region Ω_1 that have two ends on the same state circle in the state graph associated to \mathbb{H} to set up a equivalence class where we can apply Lemma 3.1. This is the primary motivation for our algorithm in the proof of Theorem 2.4.

The general case is more complicated, but we can list all the possibilities for $a(\mathbb{H})$ for the specific case we have here where $n = 3$. When $a(\mathbb{H})$ is non-empty, the possible subsets are $a(\mathbb{H}) = \{1\}, \{3\}, \{5\}, \{1, 3\}, \{1, 5\}, \{3, 5\}$, and $\{1, 3, 5\}$. In Table 1 and Table 2 we list all the possible sequences $\{b^j\}$, $\{t^j\}$ and the corresponding $G(\mathbb{H})$, $v(\mathbb{H}) - k(\mathbb{H})$, and $g(\mathbb{H})$. For example, for $a(\mathbb{H})$ to contain edges labeled 1

$a(\mathbb{H})$	$e_\ell^n \setminus a(\mathbb{H})$	$\{b^j\}$ for $R\Omega_1$	$\{t^j\}$ for $R\Omega_0$	$\{t^j\}$ for $L\Omega_0$	$\{b^j\}$ for $L\Omega_1$	$G(\mathbb{H})$	$v(\mathbb{H}) - k(\mathbb{H})$	$g(\mathbb{H})$
$\{\emptyset\}$	$\{1, 3, 5\}$	$\{\emptyset\}$	$\{\emptyset\}$	$\{\emptyset\}$	$\{\emptyset\}$	$\{\emptyset\}$	0	0
							1	0
$\{1\}$	$\{3, 5\}$	$\{\emptyset\}$	$\{\emptyset\}$	$\{\emptyset\}$	$\{\emptyset\}$	$\{\emptyset\}$	1	0
		$\{\emptyset\}$	$\{1\}$	$\{1\}$	$\{\emptyset\}$	$\{\emptyset\}$	1	1
$\{3\}$	$\{1, 5\}$	$\{R2\}$	$\{\emptyset\}$	$\{\emptyset\}$	$\{\emptyset\}$	$\{\emptyset\}$	1	0
		$\{R2, 3\}$	$\{3\}$	$\{3\}$	$\{\emptyset\}$	$\{\emptyset\}$	1	1
$\{5\}$	$\{1, 3\}$	$\{R4\}$	$\{\emptyset\}$	$\{\emptyset\}$	$\{\emptyset\}$	$\{\emptyset\}$	1	0
		$\{R4, 5\}$	$\{5\}$	$\{5\}$	$\{\emptyset\}$	$\{\emptyset\}$	1	1
$\{1, 3\}$	$\{5\}$	$\{R2\}$	$\{\emptyset\}$	$\{\emptyset\}$	$\{\emptyset\}$	$\{R2\}$	1	0
		$\{R2, 3\}$	$\{3\}$	$\{3\}$	$\{\emptyset\}$	$\{R2\}$	1	1
		$\{\emptyset\}$	$\{1\}$	$\{1\}$	$\{\emptyset\}$	$\{\emptyset\}$	1	1
				$\{1, L2\}$	$\{L2\}$	$\{\emptyset\}$	1	1
				$\{1, L2, 3\}$	$\{L2, 3\}$	$\{\emptyset\}$	1	2
		$\{R2\}$	$\{1, R2\}$	$\{1\}$	$\{\emptyset\}$	$\{\emptyset\}$	1	1
				$\{1, L2\}$	$\{L2\}$	$\{\emptyset\}$	1	1
		$\{R2, 3\}$	$\{1, R2, 3\}$	$\{1\}$	$\{\emptyset\}$	$\{\emptyset\}$	1	2
				$\{1, L2, 3\}$	$\{L2, 3\}$	$\{\emptyset\}$	1	2
		$\{R2\}$	$\{\emptyset\}$	$\{\emptyset\}$	$\{\emptyset\}$	$\{R4\}$	1	0
$\{3, 5\}$	$\{1\}$	$\{R2, 3\}$	$\{3\}$	$\{3\}$	$\{\emptyset\}$	$\{5\}$	1	1
				$\{3, L4\}$	$\{L4\}$	$\{\emptyset\}$	1	1
				$\{3, L4, 5\}$	$\{L4, 5\}$	$\{\emptyset\}$	1	2
		$\{R2, 5\}$	$\{5\}$	$\{5\}$	$\{\emptyset\}$	$\{R4\}$	1	1
		$\{R2, 3, R4\}$	$\{3, R4\}$	$\{3\}$	$\{\emptyset\}$	$\{\emptyset\}$	1	1
				$\{3, L4\}$	$\{\emptyset\}$	$\{\emptyset\}$	1	1
		$\{R2, 3, R4, 5\}$	$\{3, R4, 5\}$	$\{3\}$	$\{\emptyset\}$	$\{\emptyset\}$	1	1
				$\{3, L4, 5\}$	$\{\emptyset\}$	$\{\emptyset\}$	1	2

TABLE 1. The tables detail the various possibilities for the equivalence classes corresponding to the different cases for $a(\mathbb{H})$ and their corresponding values of $G(\mathbb{H})$, $v(\mathbb{H}) - k(\mathbb{H})$, and $g(\mathbb{H})$, when s_a is in $R\Omega_1$ or Ω_0 . The case for when s_a is in $L\Omega_1$ is analogous.

and 3, an edge attached to the portion of the state circle S_0 cut off by 1 must be included in \mathbb{H} , and one of the edges labeled 4 in $L\Omega_1$ and $R\Omega_1$ must be included. There will be various combinations for these edges that cut off state circles so that 1 ends up with two ends on different state circles in the state graph associated to \mathbb{H} , but the only two cases that we need to deal with are

- 1 is not included in \mathbb{H} .
- 1 is included in \mathbb{H} .

If 1 is not included in \mathbb{H} , the edge with label $R2$ will have two ends on the same state circle in the state graph associated to \mathbb{H} . This is the edge included in $G(\mathbb{H})$. The sequence $\{b^j\}$ for $R\Omega_1$ will either be $\{R2\}$ or $\{R2, 3\}$ since we don't include edges with labels larger than or equal to 4, and these are shown in the row starting with $a(\mathbb{H}) = \{1, 3\}$ in Table 1. On the other hand, if 1 is included, there is a corresponding increase in $g(\mathbb{H})$ from 0 to 1. Taken altogether, $v(\mathbb{H}) - k(\mathbb{H}) + g(\mathbb{H}) \geq 2$ and so we do not need to account for the contribution of those equivalence classes. This is the idea behind Lemma 3.9, which estimates the cardinality of the set $G(\mathbb{H})$ by $v(\mathbb{H}) - k(\mathbb{H}) + g(\mathbb{H})$. The more terms there are in the sequences, and so the smaller $|G(\mathbb{H})|$ is, the bigger $v(\mathbb{H}) - k(\mathbb{H}) + g(\mathbb{H})$ will be, so that we will always end up with $|G(\mathbb{H})| + v(\mathbb{H}) - k(\mathbb{H}) + g(\mathbb{H}) \geq a(\mathbb{H}) - 1$. As shown in Section 3.5, this controls the degree of the contribution of each equivalence class so that even if there is only a single element \mathbb{H} in an equivalence class, $\deg X_{\mathbb{H}} \leq M(D^n) - 4(n - 1)$.

$a(\mathbb{H})$	$e_\ell^n \setminus a(\mathbb{H})$	$\{b^j\}$ for $R\Omega_i$	$\{t^j\}$ for $R\Omega_i$	$\{t^j\}$ for $L\Omega_i$	$\{b^j\}$ for $L\Omega_i$	$G(\mathbb{H})$	$v(\mathbb{H}) - k(\mathbb{H})$	$g(\mathbb{H})$
$\{1, 3, 5\}$	$\{\emptyset\}$	$\{\emptyset\}$	$\{1\}$	$\{1\}$	$\{\emptyset\}$	$\{3, 5\}$	1	1
				$\{1, L2\}$	$\{L2\}$	$\{L4\}$	1	1
				$\{1, L4\}$	$\{L4\}$	$\{3\}$	1	1
				$\{1, L2, 3\}$	$\{L2, 3\}$	$\{5\}$	1	2
				$\{1, L2, 5\}$	$\{L2, 5\}$	$\{R4\}$	1	2
				$\{1, L4, 5\}$	$\{L4, 5\}$	$\{3\}$	1	2
				$\{1, L2, 3, L4\}$	$\{L2, 3, L4\}$	$\{\emptyset\}$	1	2
				$\{1, L2, 3, L4, 5\}$	$\{L2, 3, L4, 5\}$	$\{\emptyset\}$	1	3
		$\{R2\}$	$\{1, R2\}$	$\{1\}$	$\{\emptyset\}$	$\{R4\}$	1	1
				$\{1, L2\}$	$\{L2\}$	$\{R4, L4\}$	1	1
				$\{1, L4\}$	$\{L4\}$	$\{R4\}$	1	1
		$\{R4\}$	$\{1, R4\}$	$\{1\}$	$\{\emptyset\}$	$\{3\}$	1	1
				$\{1, L2\}$	$\{L2\}$	$\{L4\}$	1	1
				$\{1, L4\}$	$\{L4\}$	$\{3\}$	1	1
				$\{1, L2, 3\}$	$\{L2, 3\}$	$\{\emptyset\}$	1	2
				$\{1, L2, 3, L4\}$	$\{L2, 3, L4\}$	$\{\emptyset\}$	1	2
		$\{R2, 3\}$	$\{1, R2, 3\}$	$\{1\}$	$\{\emptyset\}$	$\{5\}$	1	2
				$\{1, L4\}$	$\{L4\}$	$\{\emptyset\}$	1	2
				$\{1, L2, 3\}$	$\{L2, 3\}$	$\{5\}$	1	2
				$\{1, L4, 5\}$	$\{L4, 5\}$	$\{\emptyset\}$	1	3
				$\{1, L2, 3, L4\}$	$\{L2, 3, L4\}$	$\{\emptyset\}$	1	2
				$\{1, L2, 3, L4, 5\}$	$\{L2, 3, L4, 5\}$	$\{\emptyset\}$	1	3
		$\{R2, 5\}$	$\{1, R2, 5\}$	$\{1\}$	$\{\emptyset\}$	$\{R4\}$	1	2
				$\{1, L2, 5\}$	$\{L2, 5\}$	$\{L4, R4\}$	1	2
				$\{1, L4, 5\}$	$\{L4, 5\}$	$\{R4\}$	1	2
		$\{R4, 5\}$	$\{1, R4, 5\}$	$\{1\}$	$\{\emptyset\}$	$\{3\}$	1	2
				$\{1, L2, 3\}$	$\{L2, 3\}$	$\{\emptyset\}$	1	3
				$\{1, L2, 5\}$	$\{L2, 5\}$	$\{\emptyset\}$	1	2
				$\{1, L4, 5\}$	$\{L4, 5\}$	$\{3\}$	1	2
				$\{1, L2, 3, L4, 5\}$	$\{L2, 3, L4, 5\}$	$\{\emptyset\}$	1	3
				$\{1\}$	$\{\emptyset\}$	$\{\emptyset\}$	1	2
		$\{R2, 3, R4\}$	$\{1, R2, 3, R4\}$	$\{1, L4\}$	$\{L4\}$	$\{\emptyset\}$	1	2
				$\{1, L2, 3\}$	$\{L2, 3\}$	$\{\emptyset\}$	1	2
				$\{1, L2, 3, L4\}$	$\{L2, 3, L4\}$	$\{\emptyset\}$	1	2
		$\{R2, 3, R4, 5\}$	$\{1, R2, 3, R4, 5\}$	$\{1\}$	$\{\emptyset\}$	$\{\emptyset\}$	1	3
				$\{1, L2, 3\}$	$\{L2, 3\}$	$\{\emptyset\}$	1	3
				$\{1, L4, 5\}$	$\{L4, 5\}$	$\{\emptyset\}$	1	3
				$\{1, L2, 3, L4, 5\}$	$\{L2, 3, L4, 5\}$	$\{\emptyset\}$	1	3

TABLE 2.

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